Gauss-Seidal Method

This method is a modification of Jacobi Method and sometimes it gives faster convergence. Let us consider a system of 'n' equations where $a_{ii} \neq 0$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array}\right\} \quad (1)$$

(1) can also be written as :-

$$\left.\begin{array}{l} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n) \\ \dots \\ x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{n(n-1)}x_{n-1}) \end{array}\right\} \quad (2)$$

Now let us assume first approximation $x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}$. Substituting these values into the first equation of (2), we get

$$x_1^{(2)} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)} - \dots - a_{1n}x_n^{(1)})$$

Now substituting $x_1^{(2)}, x_3^{(1)}, x_4^{(1)}, \dots, x_n^{(1)}$ into RHS of 2nd equation of (2) we get

$$x_2^{(2)} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(2)} - a_{23}x_3^{(1)} - \dots - a_{2n}x_n^{(1)})$$

In the third equation of (2) substituting the values of $x_1^{(2)}, x_2^{(2)}, x_4^{(1)}, \dots, x_n^{(1)}$ we get

$$x_3^{(2)} = \frac{1}{a_{33}}(b_3 - a_{31}x_1^{(2)} - a_{32}x_2^{(2)} - \dots - a_{3n}x_n^{(1)})$$

Similarly, in this way we get $x_n^{(2)}$ and complete the first iteration process. We repeat the entire process repeatedly till we get the values of $x_1, x_2, x_3, \dots, x_n$ to the desired degree of accuracy. Therefore, Gauss-Seidal Method is also know as method of successive displacement

Both Jacobi and Gauss-Seidal methods converge for any type of first approximation if every eqn of the system ② satisfies the condition that the sum of the absolute values of the co-efficients $\frac{a_{ij}}{a_{ii}}$ is almost equal to or in atleast one equation less than unity. That is

$$\sum_{\substack{j=1 \\ i \neq j}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \leq, i = 1, 2, 3, \dots, n$$

The sign $<$ holds good in the case of atleast one equation.

Ques Solve the following system of equations by Gauss-Seidal method:-

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Soln Given equations can also be written as

$$\left.\begin{aligned} x &= \frac{1}{27}(85 - 6y + z) \\ y &= \frac{1}{15}(72 - 6x - 2z) \\ z &= \frac{1}{54}(110 - x - y) \end{aligned}\right\} \text{①}$$

Assuming first approximation $x^{(1)}=0, y^{(1)}=0, z^{(1)}=0$
Substituting these values into the first equation of ①, we get

$$x^{(2)} = \frac{1}{27}\left(85 - 6y^{(1)} + z^{(1)}\right) = \frac{1}{27}(85) = 3.14$$

Now, substitute $x^{(2)} = 3.14, y^{(1)} = 0, z^{(1)} = 0$ into second equation of ①, we get

$$y^{(2)} = \frac{1}{15}\left(72 - 6x^{(1)} + 2z^{(1)}\right)$$
$$= \frac{1}{15}(72 - 6(3.14) - 0) = \frac{53.16}{15} = 3.54$$

Substituting $x^{(2)} = 3.14, y^{(2)} = 3.54, z^{(1)} = 0$ into third equation of ① we get

$$z^{(2)} = \frac{1}{54}\left(110 - x^{(2)} - y^{(2)}\right) = \frac{1}{54}(110 - 3.14 - 3.54)$$

$$= \frac{103.32}{54} = 1.91$$

Now to obtain third approximation we put values of $x^{(2)}, y^{(2)}$ and $z^{(2)}$ in first equation of ①, we get

$$x^{(3)} = \frac{1}{27}\left(85 - 6y^{(2)} - z^{(2)}\right) = \frac{1}{27}(85 - 6(3.54) - 1.91)$$

$$= \frac{65.67}{27} = 2.43$$

Similarly $y^{(3)} = \frac{1}{15}\left(72 - 6(x^{(3)}) - 2z^{(2)}\right)$

$$= \frac{1}{15}(72 - 6(2.43) - 2(1.91))$$

$$= \frac{53.6}{15} = 3.57$$

$$z^{(3)} = \frac{1}{54}\left(110 - x^{(3)} - y^{(3)}\right) = \frac{1}{54}(110 - 2.43 - 3.57)$$

$$= \frac{1}{54}(104) = 1.92$$

Since, these values are close to $x^{(2)}, y^{(2)}, z^{(2)}$.
Hence, $x^{(3)} = 2.43$, $y^{(3)} = 3.57$, $z^{(3)} = 1.92$ Ans